A FINITE INTERVAL IN THE SUBSEMIGROUP LATTICE OF THE FULL TRANSFORMATION MONOID

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Abstract. In this paper we describe a portion of the subsemigroup lattice of the *full transformation semigroup* Ω^Ω, which consists of all mappings on the countable infinite set Ω. Gavrilov showed that there are five maximal subsemigroups of Ω^Ω containing the symmetric group $\operatorname{Sym}(\Omega)$. The portion of the subsemigroup lattice of Ω^Ω which we describe is that between the intersection of these five maximal subsemigroups and Ω^Ω. We prove that there are only 38 subsemigroups in this interval, in contrast to the $2^{2^{\aleph_0}}$ subsemigroups between $\operatorname{Sym}(\Omega)$ and Ω^Ω.

1. Introduction

Let Ω denote an arbitrary infinite set, let Ω^Ω denote the semigroup of mappings from Ω to itself, and let $\operatorname{Sym}(\Omega)$ denote the symmetric group on Ω. The subsemigroups of Ω^Ω form an algebraic lattice with $2^{|\Omega|}$ compact (finitely generated) elements under inclusion. Pinsker and Shelah [7] proved that every alegbraic lattice with at most $2^{|\Omega|}$ compact elements can be embedded into the subsemigroup lattice of Ω^Ω. There are $2^{2^{|\Omega|}}$ distinct subsemigroups of Ω^Ω, and even this many maximal subsemigroups [1, Theorem C]. In contrast, in 1965 Gavrilov [3] showed that there are five maximal subsemigroups of Ω^Ω containing $\operatorname{Sym}(\Omega)$ when Ω is countable. In 2005, Pinsker [5] extended Gavrilov's result to sets of arbitrary infinite cardinality showing that there are $2|\alpha|+1$ such maximal subsemigroups when $|\Omega| = \aleph_\alpha$. If S is a subsemigroup of a semigroup T, then we denote this by $S \leq T$. If S and T are subsemigroups of Ω^Ω, then the *interval from T to S*, denoted $[T, S]$, is defined to be the set of proper subsemigroups of S properly containing T, i.e.

$$[T, S] = \{U \leq \Omega^\Omega : T \lneq U \lneq S\}.$$

If S is any maximal subsemigroup of Ω^Ω containing $\operatorname{Sym}(\Omega)$, as described by Gavrilov [3] and Pinsker [5], then the interval $[\operatorname{Sym}(\Omega), S]$ has cardinality 2^{2^κ} where $|\Omega| = \aleph_\alpha$ and $\kappa = \max\{\alpha, \aleph_0\}$; for further details see Pinsker [6]. The maximal subsemigroups of the maximal subsemigroups described by Gavrilov [3] are classified in [2]; perhaps surprisingly there are only countably many such semigroups. In further contrast to Pinsker and Shelah's result [7], we prove that there are only 36 subsemigroups in the interval from the intersection of the maximal subsemigroups described by Gavrilov [3] to Ω^Ω when Ω is countably infinite; we completely describe the subsemigroup lattice in this interval.

Throughout this paper, we write functions to the right of their argument and compose from left to right. If $\alpha \in \Omega$, $f \in \Omega^\Omega$ and $\Sigma \subseteq \Omega$, then $\alpha f^{-1} = \{\beta \in \Omega : \beta f = \alpha\}$, $\Sigma f = \{\alpha f : \alpha \in \Sigma\}$, and $f|_\Sigma$ denotes the *restriction* of f to Σ. If $f \in \Omega^\Omega$ and $\Sigma \subseteq \Omega$ such that $f|_\Sigma$ is injective and $\Sigma f = \Omega f$, then we will refer to Σ as a *transversal* of f.

We require the following parameters of a function $f \in \Omega^\Omega$ to describe the maximal subsemigroups containing $\operatorname{Sym}(\Omega)$ and the semigroups introduced here:

$$\begin{aligned} d(f) &= |\Omega \setminus \Omega f| \\ c(f) &= |\Omega \setminus \Sigma|, \text{ where } \Sigma \text{ is any transversal of } f \\ k(f) &= |\{\alpha \in \Omega : |\alpha f^{-1}| = \infty\}|. \end{aligned}$$

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The parameters $d(f), c(f)$, and $k(f)$ were termed the *defect*, *collapse*, and *infinite contraction index*, respectively, of f in [4]. Recall that the *kernel* of $f \in \Omega^\Omega$ is the equivalence relation:

$$\ker(f) = \{(\alpha, \beta) \in \Omega \times \Omega : \alpha f = \beta f\}.$$

Throughout the remainder of the paper we will denote by Ω an arbitrary countably infinite set. Using the notation of [1], the maximal semigroups of Ω^Ω containing $\mathrm{Sym}(\Omega)$ are:

$$\begin{aligned} S_1 =& \{f \in \Omega^\Omega : c(f) = 0 \text{ or } d(f) > 0\} \\ S_2 =& \{f \in \Omega^\Omega : c(f) > 0 \text{ or } d(f) = 0\} \\ S_3 =& \{f \in \Omega^\Omega : c(f) < \infty \text{ or } d(f) = \infty\} \\ S_4 =& \{f \in \Omega^\Omega : c(f) = \infty \text{ or } d(f) < \infty\} \\ S_5 =& \{f \in \Omega^\Omega : k(f) < \infty\}. \end{aligned}$$

For the sake of brevity, if $I \subseteq \{1, 2, 3, 4, 5\}$, then we will denote the intersection $(\bigcap_{i \in I} S_i)$ by S_I, so that $S_{1,2}$ denotes $S_1 \cap S_2$ and so on.

Main Theorem. *The interval $[S_{1,2,3,4,5}, \Omega^\Omega]$ consists of the 30 semigroups which are all possible intersections of any non-empty proper subset of $\{S_1, S_2, S_4, S_3, S_5\}$ and the semigroups:*

$$U,\ V,\ S_1 \cap U,\ S_2 \cap V,\ S_5 \cap U,\ S_5 \cap V,\ S_{1,5} \cap U,\ \textit{and } S_{2,5} \cap V,$$

where

$$\begin{aligned} U &= \{f \in \Omega^\Omega : d(f) = \infty \textit{ or } (0 < c(f) < \infty)\} \cup \mathrm{Sym}(\Omega) \\ V &= \{f \in \Omega^\Omega : c(f) = \infty \textit{ or } (0 < d(f) < \infty)\} \cup \mathrm{Sym}(\Omega). \end{aligned}$$

We will describe the relevant semigroups given in the Main Theorem, in terms of the parameters d, c, and k, in Section 2. We will show in Proposition 4.1 that U and V are semigroups. It is routine to verify that U is contained in S_3 and $S_2 \cap S_3 \leq U$. But there exists $f \in U$ such that $d(f) = \infty$ and $c(f) = 0$ and there is no such function in $S_2 \cap S_3$, and so $S_2 \cap S_3 \neq U$. Similarly, V is contained in S_4 and $S_2 \cap S_4 \lneq V$.

In Section 3 a number of technical lemmas are proved which are extensively used in the proof of the Main Theorem in Section 4.

2. Descriptions of the semigroups in the Main Theorem

In this section we describe the semigroups in the Main Theorem according to the parameters c, d, and k. We will not repeat these statements later in the paper, but have opted to collect the descriptions in one place, for ease of reference.

S	Description of the elements $f \in S$
S_1	$c(f) = 0$ or $d(f) > 0$
S_2	$c(f) > 0$ or $d(f) = 0$
S_3	$c(f) < \infty$ or $d(f) = \infty$
S_4	$c(f) = \infty$ or $d(f) < \infty$
S_5	$k(f) < \infty$
U	$d(f) = \infty$ or $(0 < c(f) < \infty)$ or $f \in \mathrm{Sym}(\Omega)$
V	$c(f) = \infty$ or $(0 < d(f) < \infty)$ or $f \in \mathrm{Sym}(\Omega)$
$S_{1,2}$	$c(f) > 0$ and $d(f) > 0$ or $f \in \mathrm{Sym}(\Omega)$
$S_{1,3}$	$d(f) = \infty$ or $c(f) = 0$ or $(0 < c(f), d(f) < \infty)$
$S_{1,4}$	$(c(f) = d(f) = \infty)$ or $(0 < d(f) < \infty)$ or $f \in \mathrm{Sym}(\Omega)$
$S_{1,5}$	$(k(f) < \infty$ and $d(f) > 0)$ or $c(f) = 0$

$S_{2,3}$	$(c(f)=d(f)=\infty)$ or $(0<c(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{2,4}$	$c(f)=\infty$ or $d(f)=0$ or $(0<c(f),d(f)<\infty)$
$S_{2,5}$	$(k(f)<\infty$ and $c(f)>0)$ or $(d(f)=0$ and $k(f)<\infty)$
$S_{3,4}$	$(c(f)=d(f)=\infty)$ or $(c(f),d(f)<\infty)$
$S_{3,5}$	$c(f)<\infty$ or $(d(f)=\infty$ and $k(f)<\infty)$
$S_{4,5}$	$(c(f)=\infty$ and $k(f)<\infty)$ or $(d(f)<\infty$ and $k(f)<\infty)$
$V\cap S_2$	$c(f)=\infty$ or $(0<c(f),d(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$U\cap S_1$	$d(f)=\infty$ or $(0<c(f),d(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$V\cap S_5$	$(k(f)<\infty$ and $c(f)=\infty)$ or $(0<d(f)<\infty$ and $k(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$U\cap S_5$	$(k(f)<\infty$ and $d(f)=\infty)$ or $(0<c(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{1,2,3}$	$(c(f)>0$ and $d(f)=\infty)$ or $(0<c(f),d(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{1,2,4}$	$(c(f)=\infty$ and $d(f)>0)$ or $(0<c(f),d(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{1,2,5}$	$(c(f),d(f)>0)$ and $k(f)<\infty$ or $f\in \mathrm{Sym}(\Omega)$
$S_{1,3,4}$	$(c(f)=d(f)=\infty)$ or $(0<d(f)<\infty$ and $c(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{1,3,5}$	$(d(f)=\infty$ and $k(f)<\infty)$ or $(c(f)<\infty$ and $d(f)>0)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{1,4,5}$	$(c(f)=d(f)=\infty$ and $k(f)<\infty)$ or $(0<d(f)<\infty$ and $k(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{2,3,4}$	$(c(f)=d(f)=\infty)$ or $(0<c(f)<\infty$ and $d(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{2,3,5}$	$(c(f)=d(f)=\infty$ and $k(f)<\infty)$ or $(0<c(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{2,4,5}$	$(c(f)=\infty$ and $k(f)<\infty)$ or $(0<c(f)<\infty$ and $d(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{3,4,5}$	$(k(f)<\infty$ and $c(f)=d(f)=\infty)$ or $(c(f),d(f)<\infty)$
$V\cap S_{2,5}$	$(k(f)<\infty$ and $c(f)=\infty)$ or $(0<c(f),d(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$U\cap S_{1,5}$	$(k(f)<\infty$ and $d(f)=\infty)$ or $(0<c(f),d(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{1,2,3,4}$	$(c(f)=d(f)=\infty)$ or $(0<c(f),d(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{1,2,3,5}$	$(k(f)<\infty, c(f)>0$ and $d(f)=\infty)$ or $(0<c(f),d(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{1,2,4,5}$	$k(f)<\infty$ and $[(c(f)=d(f)=\infty)$ or $(c(f)>0$ and $0<d(f)<\infty)]$ or $f\in \mathrm{Sym}(\Omega)$
$S_{1,3,4,5}$	$(k(f)<\infty$ and $c(f)=d(f)=\infty)$ or $(c(f)<\infty$ and $0<d(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{2,3,4,5}$	$(k(f)<\infty$ and $c(f)=d(f)=\infty)$ or $(d(f)<\infty$ and $0<c(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$
$S_{1,2,3,4,5}$	$(c(f)=d(f)=\infty$ and $k(f)<\infty)$ or $(0<c(f),d(f)<\infty)$ or $f\in \mathrm{Sym}(\Omega)$

3. Technical lemmas

We require several technical results to prove the Main Theorem, which we present in this section. We will make repeated use of the following lemma without reference.

Lemma 3.1 (Lemma 5.4 in [1])**.** *Let $u,v\in\Omega^\Omega$. Then the following hold:*

(i) $d(g)\le d(fg)\le d(f)+d(g)$;
(ii) $c(v)\le c(vu)\le c(v)+c(u)$;
(iii) *if* $c(g)<\infty=d(f)$, *then* $d(fg)=\infty$;
(iv) *if* $d(f)<\infty=c(g)$, *then* $c(fg)=\infty$;
(v) $k(fg)\le k(f)+k(g)$.

We will repeatedly use the observation that if $f,g\in\Omega^\Omega$ such that $\ker(f)=\ker(g)$ and $d(f)=d(g)$, then $f\in\langle \mathrm{Sym}(\Omega),g\rangle$. Note that if $m,n\in\mathbb{N}\cup\{\infty\}$, then there exists $f\in\Omega^\Omega$ such that $c(f)=m$ and $d(f)=n$. Also note that $f\in\Omega^\Omega$ is injective if and only if $c(f)=0$ and f is surjective if and only if $d(f)=0$.

Lemma 3.2. *Let $u, v, f \in \Omega^\Omega$. Then the following hold:*

(i) *if $0 < c(u), c(f) < \infty$ and $d(u) = d(f)$, then $f \in \langle S_{1,2,3,4,5}, u\rangle$;*
(ii) *if $c(u) < \infty$, $d(u) = \infty$, $c(v) > 0$, $d(f) = d(v) = 0$, and $0 < c(f) < \infty$, then $f \in \langle S_{1,2,3,4,5}, u, v\rangle$;*
(iii) *if u is injective, $c(f) < \infty$, and $d(u) = d(f) = \infty$, then $f \in \langle S_{1,2,3,4,5}, u\rangle$;*
(iv) *if u is injective, $d(u) > 0$, $d(v) = d(f) = \infty$, and $c(v), c(f) < \infty$, then $f \in \langle S_{1,2,3,4,5}, u, v\rangle$.*

Proof. **(i).** Let $g \in \Omega^\Omega$ be such that $\ker(g) = \ker(f)$ and such that Ωg is a transversal of u. Then $c(g) = c(f)$ and $d(g) = c(u)$ and thus $g \in S_{1,2,3,4,5}$. Since $d(gu) = d(u) = d(f)$ and $\ker(gu) = \ker(g) = \ker(f)$, $f \in \langle gu, \mathrm{Sym}(\Omega)\rangle \leq \langle u, S_{1,2,3,4,5}\rangle$.

(ii). If $c(v) < \infty$, then $f \in \langle S_{1,2,3,4,5}, v\rangle$ by Lemma 3.2(i). Suppose $c(v) = \infty$ and let $t \in S_{1,2,3,4,5}$ be such that $0 < c(t), d(t) < \infty$. Then $0 < c(tu) < \infty$ and $d(tu) = \infty$. Choose $g \in \Omega^\Omega$ such that Ωtu is a transversal of g, Ωg is a transversal of v and $k(g) < \infty$. Then $c(g) = d(g) = \infty$ and so $g \in S_{1,2,3,4,5}$. Also $0 < c(tugv) < \infty$ and $d(tugv) = 0$ and thus $f \in \langle S_{1,2,3,4,5}, tugv\rangle \leq \langle S_{1,2,3,4,5}, u, v\rangle$ by Lemma 3.2(i).

(iii). If $c(f) = 0$, then $\ker(f) = \ker(u)$ and so $f \in \langle u, S_{1,2,3,4,5}\rangle$.

Suppose $0 < c(f) < \infty$. If $g \in \Omega^\Omega$ is such that $\ker(g) = \ker(f)$ and $0 < d(g) < \infty$, then $g \in S_{1,2,3,4,5}$. But $\ker(gu) = \ker(g)$ since u is injective and so $\ker(gu) = \ker(f)$. Also $d(gu) \geq d(u) = \infty$, and so $d(gu) = d(u) = d(f)$. Thus $f \in \langle gu, \mathrm{Sym}(\Omega)\rangle \leq \langle u, S_{1,2,3,4,5}\rangle$.

(iv). If v is injective or $d(u) = \infty$, then $f \in \langle S_{1,2,3,4,5}, v\rangle$ or $f \in \langle S_{1,2,3,4,5}, u\rangle$, respectively, by part (iii). Suppose $0 < c(v)$ and $d(u) < \infty$. Let $w \in \Omega^\Omega$ be such that $c(w) = 0$ and Ωw is a transversal of v. Then $c(wv) = 0$ and $d(wv) > d(v) = \infty$, and so $f \in \langle S_{1,2,3,4,5}, wv\rangle$ by part (iii). Let $t \in \Omega^\Omega$ be any function such that Ωu is a transversal of t and $d(t) = d(w)$. Then $c(t) = d(u)$ and $0 < c(t) < \infty$. Since $d(w) = c(v)$, it follows that $0 < d(t) < \infty$. Hence $t \in S_{1,2,3,4,5}$. Since w and ut are injective, $d(ut) = d(t)$ and $d(t) = d(w)$, it follows that $w \in \langle ut, \mathrm{Sym}(\Omega)\rangle \leq \langle u, S_{1,2,3,4,5}\rangle$. To summarise, $f \in \langle S_{1,2,3,4,5}, wv\rangle \leq \langle S_{1,2,3,4,5}, u, v\rangle$, as required. □

We also require a dual of Lemma 3.2, where the values of c and d are interchanged.

Lemma 3.3. *Let $u, v, f \in \Omega^\Omega$. Then the following hold:*

(i) *if $0 < d(u), d(f) < \infty$, $c(u) = c(f)$ and $k(f) < \infty$, then $f \in \langle S_{1,2,3,4,5}, u\rangle$;*
(ii) *if $c(u) = \infty$, $d(u) < \infty$, $c(f) = c(v) = 0$, $d(v) > 0$, and $0 < d(f) < \infty$, then $f \in \langle S_{1,2,3,4,5}, u, v\rangle$;*
(iii) *if u is surjective, $d(f), k(f) < \infty$ and $c(u) = c(f) = \infty$, then $f \in \langle S_{1,2,3,4,5}, u\rangle$;*
(iv) *if u is surjective, $c(u) > 0$, $c(v) = c(f) = \infty$, $k(f) < \infty$ and $d(v), d(f) < \infty$, then $f \in \langle S_{1,2,3,4,5}, u, v\rangle$.*

Proof. **(i).** Let $g, h \in \Omega^\Omega$ be such that Ωu is a transversal of g, $d(g) = d(f)$, $\ker(h) = \ker(f)$, and Ωh is a transversal of u. Then $\ker(f) = \ker(hug)$ and $d(f) = d(hug)$. Hence $f \in \langle hug, \mathrm{Sym}(\Omega)\rangle$ and so it suffices to show that $g, h \in S_{1,2,3,4,5}$.

Since $c(g) = d(u)$ and $d(g) = d(f)$, it follows that $0 < c(g), d(g) < \infty$ and so $g \in S_{1,2,3,4,5}$. Also $c(h) = c(f)$, $d(h) = c(u)$, and $k(h) = k(f) < \infty$. So, if $c(f) = c(u) = 0$, then $h \in \mathrm{Sym}(\Omega)$. If $0 < c(f) = c(u) < \infty$, then $0 < c(h), d(h) < \infty$ and thus $h \in S_{1,2,3,4,5}$. Finally, if $c(f) = c(u) = \infty$, then $c(h) = d(h) = \infty$, $k(h) < \infty$, and so $h \in S_{1,2,3,4,5}$.

(ii). If $d(v) < \infty$, then $f \in \langle S_{1,2,3,4,5}, v\rangle$ by Lemma 3.3(i). Suppose $d(v) = \infty$. Let $g \in \Omega^\Omega$ be such that Ωv is a transversal of g, Ωg is contained in a transversal Σ of u with $0 < |\Sigma \setminus \Omega g| < \infty$, and $k(g) < \infty$. Then $c(g) = d(g) = \infty$ and so $g \in S_{1,2,3,4,5}$. Also $c(vgu) = 0$ and $0 < d(vgu) < \infty$. Therefore $f \in \langle S_{1,2,3,4,5}, vgu\rangle \leq \langle S_{1,2,3,4,5}, u, v\rangle$ by Lemma 3.3(i).

(iii). Let $g \in \Omega^\Omega$ be such that $\ker(g) = \ker(f)$ and Ωg is contained in a transversal Σ of u with $|\Sigma \setminus \Omega g| = d(f)$. Then $c(g) = d(g) = \infty$, $k(g) < \infty$ and thus $g \in S_{1,2,3,4,5}$. Also $\ker(f) = \ker(gu)$ and $d(f) = d(gu)$, and so $f \in \langle gu, \mathrm{Sym}(\Omega)\rangle \leq \langle u, S_{1,2,3,4,5}\rangle$.

(iv). If $d(v) = 0$ or $c(u) = \infty$, then the result follows from (iii). Suppose that $0 < c(u), d(v) < \infty$ and let $g \in \Omega^\Omega$ be such that Ωv is a transversal of g and Ωg is a transversal of u. Then $c(g) = d(v)$ and $d(g) = c(u)$ which implies $0 < c(g), d(g) < \infty$ and $g \in S_{1,2,3,4,5}$. Therefore $f \in \langle S_{1,2,3,4,5}, vgu\rangle$ by (iii) since $c(vgu) = \infty$, $k(vgu) < \infty$ and $d(vgu) = 0$. □

The final technical lemma we require relates to generating mappings with infinite k value, whereas the previous two lemmas are concerned with generating mappings with finite k value.

Lemma 3.4. *Let $u, v, t, f \in \Omega^\Omega$. Then the following hold:*

(i) *if $k(u) = k(f) = d(f) = \infty$, then $f \in \langle S_{1,2,3,4,5}, u\rangle$;*
(ii) *if $k(u) = k(f) = \infty$, $d(f), d(u) > 0$, $c(v) = \infty$, and $0 < d(v) < \infty$ then $f \in \langle S_{1,2,3,4,5}, u, v\rangle$;*
(iii) *if $k(u) = k(f) = \infty$, $c(v) = \infty$, and $d(v) = 0$, then $f \in \langle S_{1,2,3,4,5}, u, v\rangle$;*
(iv) *if $k(u) = k(f) = \infty$, $c(v) = \infty$, $d(v) < \infty$, $c(t) > 0$, and $d(t) = 0$, then $f \in \langle S_{1,2,3,4,5}, u, v, t\rangle$.*

Proof. **(i).** Let $\{K_i : i \in \mathbb{N}\}$ be the kernel classes of f where K_0 is infinite. Also let $\{L_i : i \in \mathbb{N}\}$ be the infinite kernel classes of u. Let $g \in \Omega^\Omega$ be such that $K_0 g = \{\alpha\}$ where $\alpha \in L_0$, $g|_{K_i}$ is injective and $K_i g \subseteq L_{2i}$. Then $c(g) = d(g) = \infty$ and $k(g) = 1$ which implies that $g \in S_{1,2,3,4,5}$. Also $\ker(gu) = \ker(f)$ and $d(gu) = d(f) = \infty$, and so $f \in \langle gu, \mathrm{Sym}(\Omega)\rangle \leq \langle u, S_{1,2,3,4,5}\rangle$.

(ii). If $d(f) = \infty$, then the result follows from (i). Suppose $0 < d(f) < \infty$ and let $g \in \Omega^\Omega$ be such that $\ker(f) = \ker(g)$ and Ωg is a transversal of v. Then $k(g) = k(f) = \infty$ and $d(g) = c(v) = \infty$ and so by (i), $g \in \langle S_{1,2,3,4,5}, u\rangle$. If $h \in \Omega^\Omega$ is such that Ωv is a transversal of h and $d(h) = d(f)$, then $0 < c(h), d(h) < \infty$ and $h \in S_{1,2,3,4,5}$. Hence $\ker(f) = \ker(gvh)$ and $d(f) = d(gvh)$, which implies that $f \in \langle g, h, v, \mathrm{Sym}(\Omega)\rangle \leq \langle u, v, S_{1,2,3,4,5}\rangle$.

(iii). Let $g \in \Omega^\Omega$ be such that $\ker(f) = \ker(g)$ and Ωg is contained in a transversal Σ of u with $|\Sigma \setminus \Omega g| = d(f)$. Then $k(g) = d(g) = \infty$ and $g \in \langle S_{1,2,3,4,5}, u\rangle$ by (i). Since $\ker(f) = \ker(gu)$ and $d(f) = d(gu)$, $f \in \langle gu, \mathrm{Sym}(\Omega)\rangle \leq \langle S_{1,2,3,4,5}, u\rangle$, as required.

(iv). If $d(v) = 0$ or $c(t) = \infty$, then $f \in \langle u, v, S_{1,2,3,4,5}\rangle$ or $f \in \langle u, t, S_{1,2,3,4,5}\rangle$, respectively, from (iii). Suppose $0 < c(t), d(v) < \infty$ and let $g \in \Omega^\Omega$ be such that Ωv is a transversal of g and Ωg is a transversal of t. Then $0 < c(g), d(g) < c(t)$ and so $g \in S_{1,2,3,4,5}$. Also $c(vgt) = \infty$ and $d(vgt) = 0$, and hence $f \in \langle S_{1,2,3,4,5}, u, vgt\rangle \leq \langle u, v, t, S_{1,2,3,4,5}\rangle$ by (iii). □

4. The Proof of the Main Theorem

In this section we prove the Main Theorem. We start by noting that since the intersection of two subsemigroups of a semigroup is a subsemigroup, it follows that the intersection of any of S_1, S_2, S_3, S_4 and S_5 is subsemigroup of Ω^Ω. In the following proposition we prove that U and V are semigroups.

Proposition 4.1. *$U = \{f \in \Omega^\Omega : d(f) = \infty$ or $0 < c(f) < \infty\} \cup \mathrm{Sym}(\Omega)$ and $V = \{f \in \Omega^\Omega : c(f) = \infty$ or $(0 < d(f) < \infty)\} \cup \mathrm{Sym}(\Omega)$ are semigroups.*

Proof. Let $f, g \in U$. If $f \in \mathrm{Sym}(\Omega)$, then $d(fg) = d(g)$ and $c(fg) = c(g)$ and so $fg \in U$. Similarly, if $g \in \mathrm{Sym}(\Omega)$, then $fg \in U$. If $d(g) = \infty$, then $d(fg) = \infty$ and $fg \in U$. If $d(f) = \infty$ and $0 < c(g) < \infty$, then by Lemma 3.1(iii), $d(fg) = \infty$ and so $fg \in U$. If $0 < c(f), c(g) < \infty$, then, by Lemma 3.1(ii), $0 < c(fg) < \infty$ and so $fg \in U$.

The proof that V is a semigroup follows by a similar argument using Lemma 3.1(i) and (iv). □

We prove the Main Theorem by finding the maximal subsemigroups in S_1, S_2, S_3, S_4 and S_5 containing $S_{1,2,3,4,5}$, then the maximal subsemigroups in each of those semigroups which contain

$S_{1,2,3,4,5}$ and so on. We give the descriptions of the maximal subsemigroups found at each stage in a separate statement, each of which can be proved using the following strategy.

Let S be a semigroup, let T be a subsemigroup of S, and let $\mathcal{M} = \{M_i : i \in I\}$ be a collection of subsemigroups of S containing T for some set I and such that $M_i \not\leq M_j$ for all $i, j \in I$ with $i \neq j$. Suppose that $\mathcal{M}$ has the following property:

> if U is a subsemigroup of S containing T and U is not contained in any M_i for any $i \in I$, then $U = S$.

Then it is routine to verify that $\mathcal{M}$ consists of the maximal subsemigroups of S containing T.

There are essentially 36 cases in the proof of the Main Theorem. However, there are 14 pairs of cases where the proof of one case can be obtained from the proof of the other by interchanging the values of c and d, interchanging any part of Lemma 3.2 by the same part of Lemma 3.3 (or vice versa), and by interchanging part (ii), (iii), or (iv) of Lemma 3.4 with part (i) of the same lemma. Therefore we will not present duplicate proofs, but will only give a proof of one of the cases in each of these pairs.

Proposition 4.2.

(i) *the maximal subsemigroups of* S_1 *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2}$, $S_{1,3}$, $S_{1,4}$ *and* $S_{1,5}$*;*
(ii) *the maximal subsemigroups of* S_2 *containing* $S_{1,2,3,4,5}$ *are* $S_{1,2}$, $S_{1,3}$, $S_{1,4}$ *and* $S_{1,5}$*;*
(iii) *the maximal subsemigroups of* S_3 *containing* $S_{1,2,3,4,5}$ *are* $S_{1,3}$, U, $S_{3,4}$ *and* $S_{3,5}$*;*
(iv) *the maximal subsemigroups of* S_4 *containing* $S_{1,2,3,4,5}$ *are:* V, $S_{2,4}$, $S_{3,4}$ *and* $S_{4,5}$*;*
(v) *the maximal subsemigroups of* S_5 *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,5}$, $S_{2,5}$, $S_{3,5}$ *and* $S_{4,5}$.

Proof. **(i).** It suffices to show that if M is any subsemigroup of S_1 containing $S_{1,2,3,4,5}$ but not contained in any of the semigroups $S_{1,2}$, $S_{1,3}$, $S_{1,4}$ and $S_{1,5}$, then $M = S_1$. Let $u_1 \in M \setminus S_{1,2}$, $u_2 \in M \setminus S_{1,3}$, $u_3 \in M \setminus S_{1,4}$ and $u_4 \in M \setminus S_{1,5}$. Then the following hold:

$$c(u_1) = 0 \text{ and } d(u_1) > 0, \qquad c(u_2) = \infty \text{ and } 0 < d(u_2) < \infty$$
$$c(u_3) < \infty \text{ and } d(u_3) = \infty, \qquad k(u_4) = \infty \text{ and } d(u_4) > 0.$$

Let $f \in S_1 \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

(1) f is injective and $0 < d(f) < \infty$ in which case by Lemma 3.3(ii), $f \in \langle S_{1,2,3,4,5}, u_1, u_2\rangle$;
(2) $0 < d(f) < \infty$, $k(f) < \infty$ and $c(f) = \infty$ in which case by Lemma 3.3(i), $f \in \langle S_{1,2,3,4,5}, u_2\rangle$;
(3) $c(f) < \infty$ and $d(f) = \infty$ in which case by Lemma 3.2(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_3\rangle$;
(4) $k(f) = \infty$ and $d(f) > 0$ in which case by Lemma 3.4(ii), $f \in \langle S_{1,2,3,4,5}, u_2, u_4\rangle$.

Therefore

$$S_1 \leq \langle S_{1,2,3,4,5}, u_1, u_2, u_3, u_4\rangle \leq M \leq S_1,$$

giving equality throughout.

(ii). Let M be any subsemigroup of S_2 containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Let $u_1 \in M \setminus S_{1,2}$, $u_2 \in M \setminus S_{2,3}$, $u_3 \in M \setminus S_{2,4}$ and $u_4 \in M \setminus S_{2,5}$. Then

$$c(u_1) > 0 \text{ and } d(u_1) = 0, \qquad c(u_2) = \infty \text{ and } d(u_2) < \infty$$
$$0 < c(u_3) < \infty \text{ and } d(u_3) = \infty, \qquad k(u_4) = \infty.$$

Let $f \in S_2 \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

(1) $0 < c(f) < \infty$ and $d(f) = 0$ in which case by Lemma 3.2(ii), $f \in \langle S_{1,2,3,4,5}, u_1, u_3\rangle$;
(2) $0 < c(f) < \infty$ and $d(f) = \infty$ in which case by Lemma 3.2(i), $f \in \langle S_{1,2,3,4,5}, u_3\rangle$;
(3) $d(f) < \infty$, $k(f) < \infty$ and $c(f) = \infty$ in which case by Lemma 3.3(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_2\rangle$;
(4) $k(f) = \infty$ in which case by Lemma 3.4(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_2, u_4\rangle$.

Therefore

$$S_2 \leq \langle S_{1,2,3,4,5}, u_1, u_2, u_3, u_4\rangle \leq M \leq S_2,$$

giving equality throughout.

(iii). The proof of this case follows by an argument analogous to that used in the proof of part (iv) as discussed before the proposition. It is also necessary in case (4) of (iv) to replace the assumption that $k(f) = \infty$ by $k(f) = \infty$ and $d(f) = \infty$ and to apply Lemma 3.4(i).

(iv). Let M be any subsemigroup of S_4 containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Let $u_1 \in M \setminus V$, $u_2 \in M \setminus S_{2,4}$, $u_3 \in M \setminus S_{3,4}$ and $u_4 \in M \setminus S_{4,5}$. Then

$$0 < c(u_1) < \infty \text{ and } d(u_1) = 0, \qquad c(u_2) = 0 \text{ and } 0 < d(u_2) < \infty$$
$$c(u_3) = \infty \text{ and } d(u_3) < \infty, \qquad k(u_4) = \infty.$$

Let $f \in S_4 \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

(1) $d(f) < \infty$, $k(f) < \infty$ and $c(f) = \infty$ in which case by Lemma 3.3(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_3\rangle$;
(2) $c(f) = 0$ and $0 < d(f) < \infty$ in which case by Lemma 3.3(i), $f \in \langle S_{1,2,3,4,5}, u_2\rangle$;
(3) $0 < c(f) < \infty$ and $d(f) = 0$ in which case by Lemma 3.2(i), $f \in \langle S_{1,2,3,4,5}, u_1\rangle$;
(4) $k(f) = \infty$ in which case by Lemma 3.4(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_3, u_4\rangle$.

Therefore

$$S_4 \leq \langle S_{1,2,3,4,5}, u_1, u_2, u_3, u_4\rangle \leq M \leq S_4,$$

giving equality throughout.

(v). Let M be any subsemigroup of S_5 containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Let $u_1 \in M \setminus S_{1,5}$, $u_2 \in M \setminus S_{2,5}$, $u_3 \in M \setminus S_{3,5}$ and $u_4 \in M \setminus S_{4,5}$. Then

$$c(u_1) > 0, k(u_1) < \infty \text{ and } d(u_1) = 0, \qquad c(u_2) = 0 \text{ and } d(u_2) > 0$$
$$c(u_3) = \infty, k(u_3) < \infty \text{ and } d(u_3) < \infty, \qquad c(u_4) < \infty \text{ and } d(u_4) = \infty.$$

Let $f \in S_5 \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

(1) $d(f) < \infty$, $k(f) < \infty$ and $c(f) = \infty$ in which case by Lemma 3.3(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_3\rangle$;
(2) $c(f) < \infty$ and $d(f) = \infty$ in which case by Lemma 3.2(iv), $f \in \langle S_{1,2,3,4,5}, u_2, u_4\rangle$;
(3) $0 < c(f) < \infty$ and $d(f) = 0$ in which case by Lemma 3.2(ii), $f \in \langle S_{1,2,3,4,5}, u_1, u_4\rangle$;
(4) $c(f) = 0$ and $0 < d(f) < \infty$ in which case by Lemma 3.3(ii), $f \in \langle S_{1,2,3,4,5}, u_2, u_3\rangle$.

Therefore

$$S_5 \leq \langle S_{1,2,3,4,5}, u_1, u_2, u_3, u_4\rangle \leq M \leq S_5,$$

giving equality throughout. □

Proposition 4.3.

(i) *the maximal subsemigroups of V containing $S_{1,2,3,4,5}$ are: $S_{1,4}$, $V \cap S_2$ and $V \cap S_5$;*
(ii) *the maximal subsemigroups of U containing $S_{1,2,3,4,5}$ are: $U \cap S_1$, $S_{2,3}$ and $U \cap S_5$.*

Proof. **(i).** Let $u_1 \in V \setminus S_{1,4}$, $u_2 \in V \setminus (V \cap S_2)$ and $u_3 \in V \setminus (V \cap S_5)$. Then

$$c(u_1) = \infty \text{ and } d(u_1) = 0, \quad c(u_2) = 0 \text{ and } 0 < d(u_2) < \infty, \quad k(u_3) = \infty.$$

Let $f \in V \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

(1) $0 < d(f) < \infty$ and $c(f) = 0$ in which case by Lemma 3.3(i), $f \in \langle S_{1,2,3,4,5}, u_2\rangle$;
(2) $d(f) < \infty$, $k(f) < \infty$ and $c(f) = \infty$ in which case by Lemma 3.3(iii), $f \in \langle S_{1,2,3,4,5}, u_1\rangle$;
(3) $k(f) = \infty$ in which case by Lemma 3.4(iii), $f \in \langle S_{1,2,3,4,5}, u_1, u_3\rangle$.

Hence if M is any subsemigroup of V containing $S_{1,2,3,4,5}$ which is not contained in any of the semigroups in the statement of the propoposition, then $M = V$.

(ii). The proof is analogous to (i). □

Proposition 4.4.

(i) *the maximal subsemigroups of $S_{1,2}$ containing $S_{1,2,3,4,5}$ are: $S_{1,2,3}$, $S_{1,2,4}$ and $S_{1,2,5}$;*
(ii) *the maximal subsemigroups of $S_{1,3}$ containing $S_{1,2,3,4,5}$ are: $U \cap S_1$, $S_{1,3,4}$ and $S_{1,3,5}$;*
(iii) *the maximal subsemigroups of $S_{1,4}$ containing $S_{1,2,3,4,5}$ are: $S_{1,2,4}$, $S_{1,3,4}$ and $S_{1,4,5}$;*
(iv) *the maximal subsemigroups of $S_{1,5}$ containing $S_{1,2,3,4,5}$ are: $S_{1,2,5}$, $S_{1,3,5}$ and $S_{1,4,5}$;*
(v) *the maximal subsemigroups of $S_{2,3}$ containing $S_{1,2,3,4,5}$ are: $S_{1,2,3}$, $S_{2,3,4}$ and $S_{2,3,5}$;*

(vi) *the maximal subsemigroups of* $S_{2,4}$ *containing* $S_{1,2,3,4,5}$ *are:* $V \cap S_2$*,* $S_{2,3,4}$ *and* $S_{2,4,5}$*;*
(vii) *the maximal subsemigroups of* $S_{2,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,5}$*,* $S_{2,3,5}$ *and* $S_{2,4,5}$*;*
(viii) *the maximal subsemigroups of* $S_{3,4}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,3,4}$*,* $S_{2,3,4}$ *and* $S_{3,4,5}$*;*
(ix) *the maximal subsemigroups of* $S_{3,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,3,5}$*,* $U \cap S_5$ *and* $S_{3,4,5}$*;*
(x) *the maximal subsemigroups of* $S_{4,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $V \cap S_5$*,* $S_{2,4,5}$ *and* $S_{3,4,5}$*.*

Proof. **(i).** Let M be any subsemigroup of $S_{1,2}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Then there exist $u_1, u_2, u_3 \in M$ such that:

$$c(u_1) = \infty \text{ and } 0 < d(u_1) < \infty, \quad 0 < c(u_2) < \infty \text{ and } d(u_2) = \infty, \quad k(u_3) = \infty \text{ and } d(u_3) > 0.$$

Let $f \in S_{1,2} \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

(1) $d(f) = \infty$ and $0 < c(f) < \infty$ in which case Lemma 3.2(i), $f \in \langle S_{1,2,3,4,5}, u_2\rangle$;
(2) $c(f) = \infty$, $k(f) < \infty$ and $0 < d(f) < \infty$ in which case Lemma 3.3(i), $f \in \langle S_{1,2,3,4,5}, u_1\rangle$;
(3) $k(f) = \infty$ and $d(f) > 0$ in which case Lemma 3.4(ii), $f \in \langle S_{1,2,3,4,5}, u_1, u_3\rangle$.

Thus $M = S_{1,2}$.

(ii). The proof is analogous to (vi).

(iii). Let M be any subsemigroup of $S_{1,4}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Then there exist $u_1, u_2, u_3 \in M$ such that:

$$c(u_1) = 0 \text{ and } 0 < d(u_1) < \infty, \quad c(u_2) = \infty \text{ and } 0 < d(u_2) < \infty, \quad k(u_3) = \infty \text{ and } d(u_3) > 0.$$

Let $f \in S_{1,4} \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

(1) $0 < d(f) < \infty$ and $c(f) = 0$ and so $f \in \langle S_{1,2,3,4,5}, u_1\rangle$ by Lemma 3.3(i);
(2) $0 < d(f) < \infty$, $k(f) < \infty$ and $c(f) = \infty$ and so $f \in \langle S_{1,2,3,4,5}, u_2\rangle$ by Lemma 3.3(i);
(3) $k(f) = \infty$ and $d(f) > 0$ and so $f \in \langle S_{1,2,3,4,5}, u_2, u_3\rangle$ by Lemma 3.4(ii).

Thus $M = S_{1,4}$, as required.

(iv). Let M be any subsemigroup of $S_{1,5}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Then there exist $u_1, u_2, u_3 \in M$ such that:

$$c(u_1) = 0 \text{ and } d(u_1) > 0$$

$$c(u_2) = \infty, \quad k(u_2) < \infty \text{ and } 0 < d(u_2) < \infty, \quad d(u_3) = \infty \text{ and } c(u_3) < \infty.$$

Let $f \in S_{1,5} \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

(1) $0 < d(f) < \infty$ and $c(f) = 0$ and so $f \in \langle S_{1,2,3,4,5}, u_1, u_2\rangle$ by Lemma 3.3(ii);
(2) $0 < d(f) < \infty$, $k(f) < \infty$ and $c(f) = \infty$ and so $f \in \langle S_{1,2,3,4,5}, u_2\rangle$ by Lemma 3.3(i);
(3) $d(f) = \infty$ and $c(f) < \infty$ and so $f \in \langle S_{1,2,3,4,5}, u_1, u_3\rangle$ by Lemma 3.2(iv).

Thus $M = S_{1,5}$, as required.

(v). The proof is analogous to (iii).

(vi). Let M be any subsemigroup of $S_{2,4}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Then there exist $u_1, u_2, u_3 \in M$ such that:

$$d(u_1) = 0 \text{ and } 0 < c(u_1) < \infty, \quad d(u_2) < \infty \text{ and } c(u_2) = \infty, \quad k(u_3) = \infty.$$

Let $f \in S_{2,4} \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

(1) $d(f) = 0$ and $0 < c(f) < \infty$ in which case Lemma 3.2(i), $f \in \langle S_{1,2,3,4,5}, u_1\rangle$;
(2) $c(f) = \infty$, $k(f) < \infty$ and $d(f) < \infty$ in which case Lemma 3.3(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_2\rangle$;
(3) $k(f) = \infty$ in which case Lemma 3.4(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_2, u_3\rangle$.

Thus $M = S_{2,4}$.

(vii). The proof is analogous to (iv).

(viii). Let M be any subsemigroup of $S_{3,4}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Then there exist $u_1, u_2, u_3 \in M$ such that:

$$0 < c(u_1) < \infty \text{ and } d(u_1) = 0, \quad c(u_2) = 0 \text{ and } 0 < d(u_2) < \infty, \quad k(u_3) = \infty \text{ and } d(u_3) = \infty.$$

Let $f \in S_{3,4} \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

(1) $0 < c(f) < \infty$ and $d(f) = 0$ and so $f \in \langle S_{1,2,3,4,5}, u_1\rangle$ by Lemma 3.2(i);
(2) $c(f) = 0$ and $0 < d(f) < \infty$ and so $f \in \langle S_{1,2,3,4,5}, u_2\rangle$ by Lemma 3.3(i);
(3) $k(f) = d(f) = \infty$ and so $f \in \langle S_{1,2,3,4,5}, u_3\rangle$ by Lemma 3.4(i).

Therefore $M = S_{3,4}$.

(ix). Let M be any subsemigroup of $S_{3,5}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Then there exist $u_1, u_2, u_3 \in M$ such that:

$$0 < c(u_1) < \infty \text{ and } d(u_1) = 0, \quad c(u_2) < \infty \text{ and } d(u_2) = \infty, \quad c(u_3) = 0 \text{ and } 0 < d(u_3) < \infty.$$

Let $f \in S_{3,5} \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

(1) $0 < c(f) < \infty$ and $d(f) = 0$ and so $f \in \langle S_{1,2,3,4,5}, u_1\rangle$ by Lemma 3.2(i);
(2) $c(f) < \infty$ and $d(f) = \infty$ and so $f \in \langle S_{1,2,3,4,5}, u_2, u_3\rangle$ by Lemma 3.2(iv);
(3) $c(f) = 0$ and $0 < d(f) < \infty$ and so $f \in \langle S_{1,2,3,4,5}, u_3\rangle$ by Lemma 3.3(i).

Whence $M = S_{3,5}$.

(x). The proof is analogous to (ix). □

Proposition 4.5.

(i) *the maximal subsemigroups of $V \cap S_2$ containing $S_{1,2,3,4,5}$ are: $S_{1,2,4}$ and $V \cap S_{2,5}$;*
(ii) *the maximal subsemigroups of $U \cap S_1$ containing $S_{1,2,3,4,5}$ are: $S_{1,2,3}$ and $U \cap S_{1,5}$;*
(iii) *the maximal subsemigroups of $V \cap S_5$ containing $S_{1,2,3,4,5}$ are: $S_{1,4,5}$ and $V \cap S_{2,5}$;*
(iv) *the maximal subsemigroups of $U \cap S_5$ containing $S_{1,2,3,4,5}$ are: $S_{2,3,5}$ and $U \cap S_{1,5}$.*

Proof. **(i).** Let $u_1 \in (V \cap S_2) \setminus S_{1,2,4}$ and let $u_2 \in (V \cap S_2) \setminus (V \cap S_{2,5})$. Then:

$$d(u_1) = 0 \text{ and } c(u_1) = \infty, \quad k(u_2) = \infty.$$

Let $f \in V \cap S_2 \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

(1) $d(f) < \infty$, $k(f) < \infty$ and $c(f) = \infty$ and so $f \in \langle S_{1,2,3,4,5}, u_1\rangle$ by Lemma 3.3(iii);
(2) $k(f) = \infty$ and so $f \in \langle S_{1,2,3,4,5}, u_1, u_2\rangle$ by Lemma 3.4(iii).

So, if M is any subsemigroup of $V \cap S_2$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = V \cap S_2$.

(ii). The proof is analogous to (i).

(iii). Let $u_1 \in (V \cap S_5) \setminus S_{1,4,5}$ and $u_2 \in (V \cap S_5) \setminus (V \cap S_{2,5})$. Then

$$c(u_1) = \infty,\ k(u_1) < \infty \text{ and } d(u_1) = 0, \quad c(u_2) = 0 \text{ and } 0 < d(u_2) < \infty.$$

If $f \in (V \cap S_5) \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

(1) $c(f) = \infty$, $k(f) < \infty$ and $d(f) < \infty$ and so Lemma 3.3(iii) implies that $f \in \langle S_{1,2,3,4,5}, u_1\rangle$;
(2) $c(f) = 0$, $0 < d(f) < \infty$, and $k(f) < \infty$ and so Lemma 3.3(i) implies that $f \in \langle S_{1,2,3,4,5}, u_2\rangle$.

So, if M is any subsemigroup of $V \cap S_5$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = V \cap S_5$.

(iv). The proof is analogous to (iii). □

Proposition 4.6.

(i) *the maximal subsemigroups of $S_{1,2,3}$ containing $S_{1,2,3,4,5}$ are: $S_{1,2,3,4}$ and $S_{1,2,3,5}$;*
(ii) *the maximal subsemigroups of $S_{1,2,4}$ containing $S_{1,2,3,4,5}$ are: $S_{1,2,3,4}$ and $S_{1,2,4,5}$;*
(iii) *the maximal subsemigroups of $S_{1,2,5}$ containing $S_{1,2,3,4,5}$ are: $S_{1,2,3,5}$ and $S_{1,2,4,5}$;*

(iv) *the maximal subsemigroups of* $S_{1,3,4}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,3,4}$ *and* $S_{1,3,4,5}$*;*
(v) *the maximal subsemigroups of* $S_{1,3,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,3,4,5}$ *and* $U \cap S_{1,5}$*;*
(vi) *the maximal subsemigroups of* $S_{1,4,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,4,5}$ *and* $S_{1,3,4,5}$*;*
(vii) *the maximal subsemigroups of* $S_{2,3,4}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,3,4}$ *and* $S_{2,3,4,5}$*;*
(viii) *the maximal subsemigroups of* $S_{2,3,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,3,5}$ *and* $S_{2,3,4,5}$*;*
(ix) *the maximal subsemigroups of* $S_{2,4,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{2,3,4,5}$ *and* $V \cap S_{2,5}$*;*
(x) *the maximal subsemigroups of* $S_{3,4,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,3,4,5}$ *and* $S_{2,3,4,5}$*.*

Proof. **(i).** Let $u_1 \in S_{1,2,3} \setminus S_{1,2,3,4}$ and let $u_2 \in S_{1,2,3} \setminus (U \cap S_{1,5})$. Then

$$0 < c(u_1) < \infty \text{ and } d(u_1) = \infty, \quad k(u_2) = d(u_2) = \infty.$$

If $f \in S_{1,2,3} \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

(1) $0 < c(f) < \infty$ and $d(f) = \infty$ and so $f \in \langle u_1, S_{1,2,3,4,5}\rangle$ by Lemma 3.2(i);
(2) $k(f) = d(f) = \infty$ and so $f \in \langle u_2, S_{1,2,3,4,5}\rangle$ by Lemma 3.4(i).

So, if M is any subsemigroup of $S_{1,2,3}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = S_{1,2,3}$.

(ii). Let $u_1 \in S_{1,2,4} \setminus S_{1,2,3,4}$ and $u_2 \in S_{1,2,4} \setminus S_{1,2,4,5}$. Then

$$c(u_1) = \infty,\ k(u_1) < \infty \text{ and } 0 < d(u_1) < \infty, \quad k(u_2) = \infty \text{ and } d(u_2) > 0.$$

If $f \in S_{1,2,4} \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

(1) $c(f) = \infty$, $k(f) < \infty$ and $0 < d(f) < \infty$ and so $f \in \langle u_1, S_{1,2,3,4,5}\rangle$ by Lemma 3.3(i);
(2) $k(f) = \infty$ and $d(f) > 0$ and so $f \in \langle u_1, u_2, S_{1,2,3,4,5}\rangle$ by Lemma 3.4(ii).

So, if M is any subsemigroup of $S_{1,2,4}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = S_{1,2,4}$.

(iii). Let $u_1 \in S_{1,2,5} \setminus S_{1,2,3,5}$ and $u_2 \in S_{1,2,5} \setminus S_{1,2,4,5}$. Then

$$c(u_1) = \infty,\ k(u_1) < \infty \text{ and } 0 < d(u_1) < \infty, \quad 0 < c(u_2) < \infty \text{ and } d(u_2) = \infty.$$

If $f \in S_{1,2,5} \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

(1) $0 < c(f) < \infty$ and $d(f) = \infty$ and so $f \in \langle u_2, S_{1,2,3,4,5}\rangle$ by Lemma 3.2(i);
(2) $c(f) = \infty$, $k(f) < \infty$ and $0 < d(f) < \infty$ and so $f \in \langle u_1, S_{1,2,3,4,5}\rangle$ by Lemma 3.3(i).

So, if M is any subsemigroup of $S_{1,2,5}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = S_{1,2,5}$.

(iv). Let $u_1 \in S_{1,3,4} \setminus S_{1,2,3,4}$ and $u_2 \in S_{1,3,4} \setminus S_{1,3,4,5}$. Then

$$c(u_1) = 0 \text{ and } 0 < d(u_1) < \infty, \quad k(u_2) = d(u_2) = \infty.$$

If $f \in S_{1,3,4} \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

(1) $c(f) = 0$ and $0 < d(f) < \infty$ and so $f \in \langle u_1, S_{1,2,3,4,5}\rangle$ by Lemma 3.3(i);
(2) $k(f) = d(f) = \infty$ and so $f \in \langle u_2, S_{1,2,3,4,5}\rangle$ by Lemma 3.4(i).

So, if M is any subsemigroup of $S_{1,3,4}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = S_{1,3,4}$.

(v). Let $u_1 \in S_{1,3,5} \setminus (U \cap S_{1,5})$ and $u_2 \in S_{1,3,5} \setminus S_{1,3,4,5}$. Then

$$c(u_1) = 0 \text{ and } 0 < d(u_1) < \infty, \quad c(u_2) < \infty \text{ and } d(u_2) = \infty.$$

If $f \in S_{1,3,5} \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

(1) $c(f) = 0$ and $0 < d(f) < \infty$ and so $f \in \langle u_1, S_{1,2,3,4,5}\rangle$ by Lemma 3.3(i);
(2) $c(f) < \infty$ and $d(f) = \infty$ and so $f \in \langle u_1, u_2, S_{1,2,3,4,5}\rangle$ by Lemma 3.2(iv).

So, if M is any subsemigroup of $S_{1,3,5}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = S_{1,3,5}$.

(vi). The proof is analogous to (viii).

(vii). The proof is analogous to (iv).

(viii). Let $u_1 \in S_{2,3,5} \setminus S_{1,2,3,5}$ and $u_2 \in S_{2,3,5} \setminus S_{2,3,4,5}$. Then

$$0 < c(u_1) < \infty \text{ and } d(u_1) = 0, \quad 0 < c(u_2) < \infty \text{ and } d(u_2) = \infty.$$

If $f \in S_{2,3,5} \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

(1) $0 < c(f) < \infty$ and $d(f) = 0$ and so $f \in \langle u_1, S_{1,2,3,4,5}\rangle$ by Lemma 3.2(i);
(2) $0 < c(f) < \infty$ and $d(f) = \infty$ and so $f \in \langle u_2, S_{1,2,3,4,5}\rangle$ by Lemma 3.2(i).

So, if M is any subsemigroup of $S_{2,3,5}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = S_{2,3,5}$.

(ix). The proof is analogous to (v).

(x). Let $u_1 \in S_{3,4,5} \setminus S_{1,3,4,5}$ and let $u_2 \in S_{3,4,5} \setminus S_{2,3,4,5}$. Then

$$0 < c(u_1) < \infty \text{ and } d(u_1) = 0, \quad c(u_2) = 0 \text{ and } 0 < d(u_2) < \infty.$$

If $f \in S_{3,4,5} \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

(1) $0 < c(f) < \infty$ and $d(f) = 0$ and $f \in \langle u_1, S_{1,2,3,4,5}\rangle$ by Lemma 3.2(i);
(2) $c(f) = 0$ and $0 < d(f) < \infty$ and so $f \in \langle u_2, S_{1,2,3,4,5}\rangle$ by Lemma 3.3(i).

So, if M is any subsemigroup of $S_{3,4,5}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = S_{3,4,5}$. □

Proposition 4.7.

(i) *the only maximal subsemigroup of* $V \cap S_{2,5}$ *containing* $S_{1,2,3,4,5}$ *is* $S_{1,2,4,5}$.
(ii) *the only maximal subsemigroup of* $U \cap S_{1,5}$ *containing* $S_{1,2,3,4,5}$ *is* $S_{1,2,3,5}$.

Proof. **(i)** The semigroup $S_{1,2,4,5}$ is described in Proposition 4.6(ii). Let $u \in (V \cap S_{2,5}) \setminus S_{1,2,4,5}$. Then $c(u) = \infty$ and $d(u) = 0$ (also $k(u) < \infty$ holds) and so by Lemma 3.3(iii)

$$V \cap S_{2,5} = S_{1,2,3,4,5} \cup \{f \in \Omega^\Omega : d(f) < \infty,\ k(f) < \infty, \text{ and } c(f) = \infty\} \subseteq \langle u, S_{1,2,3,4,5}\rangle \subseteq \langle u, S_{1,2,4,5}\rangle.$$

Therefore if M is any subsemigroup of $V \cap S_{2,5}$ containing $S_{1,2,3,4,5}$ which is not contained in $S_{1,2,4,5}$, then $M = V \cap S_{2,5}$.

(ii) The proof is analogous to (i). □

Proposition 4.8. $S_{1,2,3,4,5}$ *is maximal in* $S_{1,2,3,4}$, $S_{1,2,3,5}$, $S_{1,2,4,5}$, $S_{1,3,4,5}$, *and* $S_{2,3,4,5}$.

Proof. If $u \in S_{1,2,3,4} \setminus S_{1,2,3,4,5}$, then $k(u) = d(u) = \infty$. Hence by Lemma 3.4(i)

$$\langle u, S_{1,2,3,4,5}\rangle \supseteq \{f \in \Omega^\Omega : k(f) = d(f) = \infty\} \cup S_{1,2,3,4,5} = S_{1,2,3,4},$$

and so $S_{1,2,3,4,5}$ is maximal in $S_{1,2,3,4}$.

If $u \in S_{1,2,3,5} \setminus S_{1,2,3,4,5}$, then $0 < c(u) < \infty$ and $d(u) = \infty$. Therefore by Lemma 3.2(i)

$$\langle u, S_{1,2,3,4,5}\rangle \supseteq \{f \in \Omega^\Omega : 0 < c(f) < \infty \text{ and } d(f) = \infty\} \cup S_{1,2,3,4,5} = S_{1,2,3,5}$$

and so $S_{1,2,3,4,5}$ is maximal in $S_{1,2,3,5}$. The proof that $S_{1,2,3,4,5}$ is maximal in $S_{1,2,4,5}$ is analogous to the proof that $S_{1,2,3,4,5}$ is maximal in $S_{1,2,3,5}$ using Lemma 3.3(i).

If $u \in S_{1,3,4,5} \setminus S_{1,2,3,4,5}$, then $c(u) = 0$ and $0 < d(u) < \infty$. Thus by Lemma 3.3(i)

$$\langle u, S_{1,2,3,4,5}\rangle \supseteq \{f \in \Omega^\Omega : c(f) = 0 \text{ and } 0 < d(f) < \infty\} \cup S_{1,2,3,4,5} = S_{1,3,4,5}.$$

In particular, $S_{1,2,3,4,5}$ is maximal in $S_{1,3,4,5}$. The proof that $S_{1,2,3,4,5}$ is maximal in $S_{2,3,4,5}$ is analogous to the proof that $S_{1,2,3,4,5}$ is maximal in $S_{1,3,4,5}$ using Lemma 3.2(i). □

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